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Asymmetric creeping motion of an open torus

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This paper presents exact solutions using toroidal co-ordinates to the equations of creeping fluid motion with the no-slip boundary conditions for a toroidal particle translating in a direction normal to the axis of symmetry or rotating about an axis normal to the axis of symmetry through an otherwise infinite expanse of quiescent fluid. The associated resisting force and resisting torque are computed for toroids of various geometrical ratios b/a, b being the smallest radius of the open hole and $(b+2a)$ being the radius to the outermost rim of the torus. These results are compared with approximate calculations based on slender-body theory and on the theory for interacting beads. The exact and approximate calculations become asymptotically equal as b/a becomes very large, but departures from the exact calculations are apparent for b/a less than 10–100 depending on the mode of motion and the method of approximation and the approximations are unreliable for b/a less than 2·0.

1. Introduction

This paper presents exact solutions to the equations of creeping fluid motion with the no-slip boundary conditions for a toroidal particle translating in a direction normal to the axis of symmetry or rotating about an axis normal to the axis of symmetry through an otherwise infinite expanse of quiescent fluid. The associated resisting force and resisting torque are computed for toroids of various geometrical ratios b/a, b being the smallest radius of the open hole and $(b+2a)$ being the radius to the outermost rim of the torus; see figure 1. In the case of a translating torus, no net torque is exerted on the body by the fluid and likewise when the torus rotates without translation, it does not experience a net force. These solutions supplement the exact solution of Kanwal (1961) for rotation about the symmetry axis and that of Payne & Pell (1960) and Majumdar & O'Neill (1977) for translation along the symmetry axis, so that the arbitrary low-speed motion of a rigid torus through a quiescent fluid can now be constructed by appropriate superposition of these four solutions.

Several approximate calculations accurate for large b/a can be found in the literature, and we shall compare our exact calculations with these approximations. Tchen (1954) studied the motion of 'skew shaped' particles, a limiting case of which are toroids. He used Burgers' (1938) method of regarding the fluid motion to be that induced by a continuous distribution of force along a curve through the particle.

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The boundary conditions at the surface of the particle require the force distribution to satisfy an integral equation whose solution permits calculation of the net resisting forces and torques. Wu (1977) reports the results of an analysis of Johnson (1976) who used slender-body theory to resolve the fluid motion into fundamental flow singularities (Stokeslet, Rotlet, etc.) distributed continuously along a circle through the torque. The calculations of Tchen and Johnson are closely related, and resulting formulae for the resistance to motion along the symmetry axis and rotation about the symmetry axis are virtually identical for the two analyses. Tchen's calculation for asymmetric translation appears to have a serious error and he does not give results for asymmetric rotation.

Yamakawa & Yamaki (1973) using methods of Kirkwood & Riseman (1948) studied the motion of a rigid array of touching beads placed at the corners of a planar polygon. Hydrodynamic interactions between the beads are allowed for through the use of the unmodified Oseen tensor and (it is to be hoped) somewhat more accurately through the use of the modified Oseen tensor. For polygons containing many uniform beads whose radii are small compared to the radius of the polygon and whose hydrodynamic interaction is limited to the unmodified Oseen tensor, Yamakawa & Yamaki developed relatively simple asymptotic expressions for the resisting forces and torques. For small numbers of beads complicated sums must be computed. Bloomfield (1976), in private correspondence to Professor C. T. O'Konski, has computed these sums for polygons consisting of from three to ten beads for both the unmodified and modified Oseen tensor to give the resisting torques for rotation about the axis of symmetry and about a transverse axis.

Our interest in seeking exact solutions for the motion of toroidal particles partly stems from the indication that some important biological macromolecules are approximately toroidal in shape. Examples are low molecular weight circular DNA and acetylcholine receptor; the latter is believed to play a crucial role in the transmission of nerve signals. These molecules are under active investigation by biochemists. Among the experimental techniques employed are electro-optic birefringence relaxation which gives information about the rotary diffusion coefficient and dynamic or quasi-elastic light scattering which gives information about the translational diffusion coefficient. To interpret these measurements accurate values of the rotary and translational frictional coefficients for small and moderate values of b/a are needed. Another possible interesting application is provided by the claim of Israelachvili, Mitchell & Ninham (1976) that some micelles are toroidal.

Beyond the application of these calculations to special particles, exact values for frictional coefficients may be of considerable value in evaluating the accuracy of the Kirkwood & Riseman bead calculations which have been used extensively in macromolecular hydrodynamics.

2. Translation of a torus along a transverse axis

A rigid open torus of geometrical ratio b/a, b being the smallest radius of the open hole and $(b+2a)$ being the radius to the outermost rim of the torus, moves through an infinite, homogeneous, incompressible fluid of density ρ and viscosity μ which is at rest at an infinite distance from the torus. In a system of Cartesian co-ordinates

(x, y, z) in which z is the axis of symmetry and the plane $z = 0$ is a plane of symmetry of the body, we shall suppose that the torus translates with velocity U along the x axis (see figure 1).

We assume that the Reynolds numbers $Ua\rho/\mu$ and $Ub\rho/\mu$ are sufficiently small to allow us to neglect the nonlinear inertia terms in the Navier–Stokes equations of fluid motion. Accordingly the equations governing the flow are the Stokes' creeping-flow equation

$$\nabla p = \mu \nabla^2 \mathbf{v} \tag{1}$$

and the equation of continuity

$$\nabla . \mathbf{v} = 0 \tag{2}$$

where $\mathbf{v}$ and p are respectively the fluid velocity and pressure fields.

At infinity the fluid velocity $\mathbf{v}$ vanishes. The boundary conditions on the surface of the torus are conveniently expressed in terms of cylindrical polar co-ordinates $(\tilde{\omega}, \theta, z)$ with corresponding velocities (u, v, w) which are related to the Cartesian values in the ordinary way. For the case of translation with velocity U along the x axis, the no-slip boundary condition requires that on the particle's surface $\mathbf{v} = U\mathbf{i}_x$, or equivalently

$$u = U\cos\theta, \quad v = -U\sin\theta, \quad w = 0 \quad \text{(translation in } x \text{ direction).} \tag{3}$$

If $\mathbf{v}$ satisfies equations (1) and (2), then the pressure p and the vector $\mathbf{V} \equiv \mathbf{v} - \mathbf{r}p/2\mu$ both satisfy Laplace's equation. Guided by this fact, by the dependence of the velocity components on θ required by the boundary conditions, by the work of Majumdar & O'Neill (1977) for the axisymmetric translation of a torus, by the analysis of Dean & O'Neill (1963) for the rotation of a sphere near a plane wall about an axis parallel to the wall, and by the analysis of O'Neill (1964) for the translation of a sphere near a wall in a direction parallel to the wall, we seek solutions for the translational problem of the following form:

$$\left.\begin{aligned} p &= (\mu U/c)\tilde{Q}\cos\theta, \\ u &= \tfrac{1}{2}U\{\tilde{U} + \tilde{V} + \tilde{\omega}\tilde{Q}/c\}\cos\theta, \\ v &= \tfrac{1}{2}U\{\tilde{U} - \tilde{V}\}\sin\theta, \\ w &= \tfrac{1}{2}U\{2\tilde{W} + z\tilde{Q}/c\}\cos\theta. \end{aligned}\right\} \tag{4}$$

The factor c is a constant with the dimension of length whose significance will be apparent. Equations (4) introduce four new functions of $\tilde{\omega}$ and z, namely $\tilde{Q}$, $\tilde{U}$, $\tilde{V}$ and $\tilde{W}$ to be found in place of p, u, v and w. Substitution of (4) into (1) shows that these new functions must satisfy the differential equations

$$L_1^2 \tilde{Q} = L_2^2 \tilde{U} = L_0^2 \tilde{V} = L_1^2 \tilde{W} = 0 \tag{5}$$

where

$$L_m^2 \equiv \frac{\partial^2}{\partial\tilde{\omega}^2} + \frac{1}{\tilde{\omega}}\frac{\partial}{\partial\tilde{\omega}} - \frac{m^2}{\tilde{\omega}^2} + \frac{\partial^2}{\partial z^2}. \tag{6}$$

Furthermore, substitution of (4) into the continuity equation shows that

$$\left(3 + \tilde{\omega}\frac{\partial\tilde{Q}}{\partial\tilde{\omega}} + z\frac{\partial\tilde{Q}}{\partial z}\right) + c\left(\frac{\partial\tilde{U}}{\partial\tilde{\omega}} + \frac{2\tilde{U}}{\tilde{\omega}}\right) + c\frac{\partial\tilde{V}}{\partial\tilde{\omega}} + 2c\frac{\partial\tilde{W}}{\partial z} = 0. \tag{7}$$

When equations (4) are substituted into (3), the boundary conditions for a translating particle, we find after a little rearrangement the following conditions to be satisfied on the surface of the particle:

$$\tilde{U} - \tilde{V} = -2, \quad z\tilde{U} - \tilde{\omega}\tilde{W} = 0, \quad 2\tilde{U} + \tilde{\omega}\tilde{Q}/c = 0. \tag{8}$$

To solve the boundary-value problem expressed in (5) to (8), we introduce toroidal co-ordinates (ξ, η) which are related to the cylindrical polar co-ordinates by the formulae

$$\tilde{\omega} = \frac{c\sinh\eta}{(\cosh\eta - \cos\xi)}, \quad z = \frac{c\sin\xi}{(\cosh\eta - \cos\xi)}. \tag{9}$$

The surface $\eta = \eta_0$ is a torus with circular cross-section of radius $a = c\,\mathrm{cosech}\,\eta_0$, the centre of the cross-section in any azimuthal plane being at a distance $(a+b) = c\coth\eta_0$ from the z axis. With a and b given, c and η_0 are uniquely determined; in particular $\cosh\eta_0 = 1 + b/a$. The region occupied by the fluid is defined by $0 \leqslant \xi \leqslant 2\pi$, $0 \leqslant \theta \leqslant 2\pi$ and $0 \leqslant \eta < \eta_0$. The origin $\tilde{\omega} = z = 0$ corresponds to $\xi = \pi, \eta = 0$ while infinity is given by $\xi = 0, \eta = 0$.

For translation of the toroid in the x direction, u and v are clearly even functions of z whereas w is an odd function of z. Therefore by virtue of equations (9) and (4), $\tilde{Q}$, $\tilde{U}$ and $\tilde{V}$ must be even functions of ξ about $\xi = \pi$ while $\tilde{W}$ is an odd function of ξ about $\xi = \pi$. From Hobson (1965) we find that solutions to (5) in toroidal co-ordinates for $\tilde{Q}$, $\tilde{U}$, $\tilde{V}$ and $\tilde{W}$ with the appropriate symmetry are

$$\left.\begin{aligned}
\tilde{W} &= (\cosh\eta - \cos\xi)^{\frac{1}{2}} \sum_{n=1}^{\infty} A_n P^1_{n-\frac{1}{2}}(\cosh\eta)\sin n\xi,\\
\tilde{Q} &= (\cosh\eta - \cos\xi)^{\frac{1}{2}} \sum_{n=0}^{\infty} B_n P^1_{n-\frac{1}{2}}(\cosh\eta)\cos n\xi,\\
\tilde{V} &= (\cosh\eta - \cos\xi)^{\frac{1}{2}} \sum_{n=0}^{\infty} C_n P_{n-\frac{1}{2}}(\cosh\eta)\cos n\xi,\\
\tilde{U} &= (\cosh\eta - \cos\xi)^{\frac{1}{2}} \sum_{n=0}^{\infty} D_n P^2_{n-\frac{1}{2}}(\cosh\eta)\cos n\xi,
\end{aligned}\right\} \tag{10}$$

where the coefficients A_n, B_n, C_n and D_n are independent of ξ and η and $P^m_{n-\frac{1}{2}}$ denotes the associated Legendre function of the first kind of order $n-\frac{1}{2}$ and degree m. It will be noticed that these forms for $\tilde{Q}$, $\tilde{U}$, $\tilde{V}$ and $\tilde{W}$ automatically satisfy the boundary condition of vanishing velocity at infinity.

On using the expansion, given in Hobson (1965),

$$(\cosh\eta - \cos\xi)^{-\frac{1}{2}} = \frac{2\sqrt{2}}{\pi}\left\{\tfrac{1}{2}Q_{-\frac{1}{2}}(\cosh\eta) + \sum_{n=1}^{\infty} Q_{n-\frac{1}{2}}(\cosh\eta)\cos n\xi\right\} \tag{11}$$

where $Q_{n-\frac{1}{2}}$ denotes the associated Legendre function of the second kind of order $n-\frac{1}{2}$, and on substituting the series expansions into the three boundary conditions as expressed by (8), we find

$$C_n P_{n-\frac{1}{2}} = D_n P^2_{n-\frac{1}{2}} + \frac{4\sqrt{2}}{\pi} Q_{n-\frac{1}{2}}\epsilon_n \quad (n \geqslant 0),$$

$$A_n 2\sinh\eta_0 P^1_{n-\frac{1}{2}} = D_{n-1}P^2_{n-\frac{3}{2}}\delta_n - D_{n+1}P^2_{n+\frac{1}{2}} \quad (n \geqslant 1) \tag{12}$$

and

$$B_n \sinh \eta_0 P^1_{n-\frac{1}{2}} = D_{n-1} P^2_{n-\frac{3}{2}} \delta_n - D_n 2 \cosh \eta_0 P^2_{n-\frac{1}{2}} + D_{n+1} P^2_{n+\frac{1}{2}} \quad (n \geqslant 0),$$

where

$$\epsilon_0 = \tfrac{1}{2} \quad \text{and} \quad \epsilon_n = 1 \quad \text{for} \quad n \geqslant 1, \tag{13}$$

and

$$\delta_0 = 0, \quad \delta_1 = 2 \quad \text{and} \quad \delta_n = 1 \quad \text{for} \quad n \geqslant 2,$$

and the argument of all the Legendre functions in these equations is $\cosh \eta_0$. A fourth relationship between the coefficients A_n, B_n, C_n and D_n is required. This is provided by the continuity equation as expressed in equation (7). When the series expansions for $\tilde{Q}$, $\tilde{U}$, $\tilde{V}$ and $\tilde{W}$ are substituted into (7), we find after a great deal of manipulation:

$$\begin{aligned} &A_{n-1}(n-\tfrac{3}{2})\,\zeta_n - A_n\, 2n + A_{n+1}(n+\tfrac{3}{2}) - B_{n-1}(\tfrac{1}{2})\,(n-\tfrac{3}{2})\,\delta_n \\ &\quad + B_n(\tfrac{5}{2}) + B_{n+1}(\tfrac{1}{2})\,(n+\tfrac{3}{2}) - C_{n-1}(\tfrac{1}{2})\,\delta_n + C_n - (\tfrac{1}{2})\,C_{n+1} \\ &\quad - D_{n-1}(\tfrac{1}{2})\,(n-\tfrac{5}{2})\,(n-\tfrac{3}{2})\,\delta_n + D_n(n-\tfrac{3}{2})\,(n+\tfrac{3}{2}) \\ &\quad - D_{n+1}(\tfrac{1}{2})\,(n+\tfrac{3}{2})\,(n+\tfrac{5}{2}) = 0 \end{aligned} \tag{14}$$

for $n \geqslant 0$ where

$$\zeta_0 = 0, \quad \zeta_1 = 0 \quad \text{and} \quad \zeta_n = 1 \quad \text{for} \quad n \geqslant 2. \tag{15}$$

On substituting (12) for A_n, B_n and C_n in (14) we obtain a second-order difference equation for the coefficients D_n, namely

$$\begin{aligned} &D_{n-1}\delta_n\{(5-2n)\,X_n + (\tfrac{1}{2})\,(n-\tfrac{3}{2})\,W_{n-1} - (\tfrac{1}{2})\,Z_{n-1} - (\tfrac{1}{2})\,(n-\tfrac{5}{2})\,(n-\tfrac{3}{2})\} \\ &\quad + D_n\{(2n+3)\,X_{n+1}\delta_{n+1} - (n-\tfrac{3}{2})\,Y_{n-1}(\zeta_n+\delta_n) - \tfrac{5}{2}W_n + Z_n + (n-\tfrac{3}{2})\,(n+\tfrac{3}{2})\} \\ &\quad + D_{n+1}\{(5+2n)\,Y_n - \tfrac{1}{2}(n+\tfrac{3}{2})\,W_{n+1} - \tfrac{1}{2}Z_{n+1} - (\tfrac{1}{2})\,(n+\tfrac{3}{2})\,(n+\tfrac{5}{2})\} \\ &= (\tfrac{1}{2})\,V_{n-1}\epsilon_{n-1}\delta_n - V_n\epsilon_n + (\tfrac{1}{2})\,V_{n+1}\epsilon_{n+1}, \end{aligned} \tag{16}$$

for $n \geqslant 0$ where ϵ_n, δ_n and ζ_n are as defined above and

$$\left.\begin{aligned} V_n &\equiv 4\sqrt{2}Q_{n-\frac{1}{2}}/\pi P_{n-\frac{1}{2}}, \\ W_n &\equiv 4\cosh\eta_0 P^2_{n-\frac{1}{2}}/2\sinh\eta_0 P^1_{n-\frac{1}{2}}, \\ X_n &\equiv P^2_{n-\frac{3}{2}}/2\sinh\eta_0 P^1_{n-\frac{1}{2}}, \\ Y_n &\equiv P^2_{n+\frac{1}{2}}/2\sinh\eta_0 P^1_{n-\frac{1}{2}}, \\ Z_n &\equiv P^2_{n-\frac{1}{2}}/P_{n-\frac{1}{2}}, \end{aligned}\right\} \tag{17}$$

and again the argument of all the Legendre functions is $\cosh \eta_0$.

The solution D_n we seek must converge to zero as n becomes large and this permits the required solution to be found by successive truncation to a finite set of linear equations. Once the values of D_n are known, A_n, B_n and C_n are readily computed from (12). In carrying out these numerical calculations sufficient terms have been retained for each value of b/a so that the calculated forces are correct to at least four significant figures.

The force $\mathbf{F}$ acting on the torus is given by

$$\mathbf{F} = \int_s \mathbf{R}_n . d\mathbf{S} \tag{18}$$

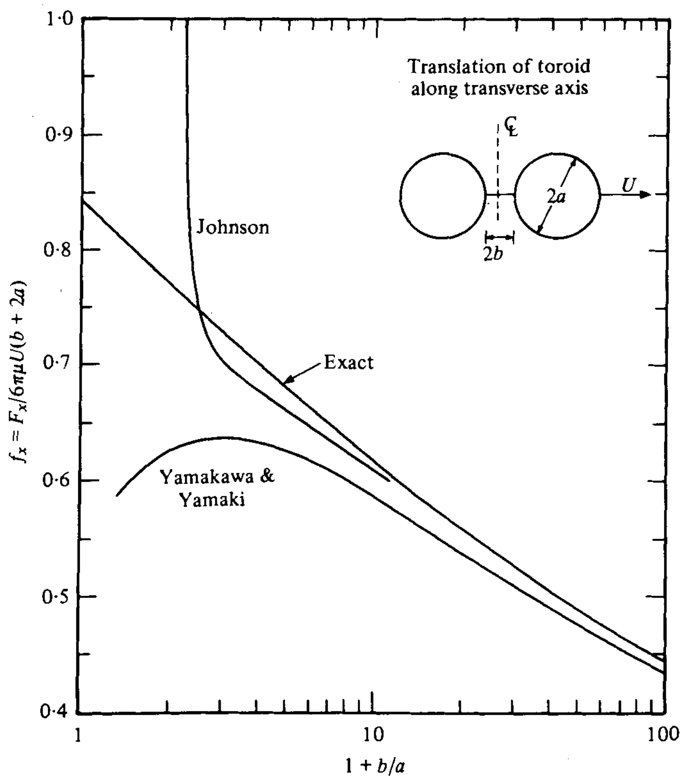

FIGURE 1. Comparison of exact and approximate dimensionless force coefficients for translation of a torus along a transverse axis.

where $\mathbf{R}_n$ is the stress vector associated with the direction $\mathbf{n}$ of the outward normal at any point on the surface of the torus. Substituting the derived expressions for the pressure and velocity gradients evaluated at the surface of the torus and carrying out the indicated integration with aid of integral identities of the type

$$\int_0^{2\pi} \cos n\xi\,(\cosh\eta - \cos\xi)^{-\frac{1}{2}}\,d\xi = 2\sqrt{2}Q_{n-\frac{1}{2}}(\cosh\eta)$$

which are derivable from the Fourier cosine expansion in (11) and its derivatives with respect to η, we obtain

$$F_x = \tfrac{1}{2}\sqrt{2}\pi\mu Uc \sum_{n=0}^{\infty} [4C_n + (4n^2-1)B_n]. \tag{19}$$

The details are not given here because it is possible to derive expressions for the force much more simply from the far-field expressions for the fluid velocity. At a great distance from the torus the flow field must approach that associated with application of a point force F_x in the x direction applied at the origin. This flow field is given by Happel & Brenner (1965) in Cartesian co-ordinates as $(F_x/8\pi\mu r^3)\,[(r^2+x^2), xy, xz]$

$1+b/a$	$f_x = f_y$	f_z	$g_x = g_y$	g_z
1·01	0·8422	0·9351	0·6485	0·7961
1·1	0·8340	0·9318	0·6362	0·7851
1·2	0·8255	0·9286	0·6238	0·7742
1·4	0·8100	0·9227	0·6027	0·7545
1·6	0·7964	0·9172	0·5855	0·7371
1·8	0·7842	0·9121	0·5711	0·7217
2	0·7733	0·9072	0·5590	0·7078
3	0·7318	0·8846	0·5184	0·6550
4	0·7032	0·8647	0·4943	0·6189
5	0·6817	0·8471	0·4776	0·5920
6	0·6644	0·8316	0·4646	0·5708
8	0·6377	0·8055	0·4451	0·5388
10	0·6174	0·7843	0·4304	0·5150
20	0·5580	0·7168	0·3857	0·4473
40	0·5045	0·6523	0·3435	0·3888
60	0·4764	0·6175	0·3206	0·3589
80	0·4579	0·5944	0·3055	0·3397
100	0·4442	0·5773	0·2943	0·3257

Note: Calculations of Dorrepaal *et al.* (1976) for the *axisymmetrical* motions of a closed torus give $f_z = 0{\cdot}9353$ and $g_z = 0{\cdot}7969$, while calculations of Majumdar & O'Neill (1979) for the *asymmetrical* motions give $f_x = f_y = 0{\cdot}8434$ and $g_x = g_y = 0{\cdot}6498$ when $b/a = 0$.

TABLE 1. Values of the dimensionless force and torque coefficients for toroids of various geometry

where $r^2 = x^2+y^2+z^2 = \tilde{\omega}^2+z^2$. Thus F_x must also be given by the following three limits as $r \to \infty$, i.e. ξ and $\eta \to 0$,

$$\left.\begin{aligned} F_x &= \lim_{r\to\infty}\{8\pi\mu r^3 w/xz\},\\ F_x &= \lim_{r\to\infty}\{8\pi\mu r^3(u\sin\theta+v\cos\theta)/xy\},\\ F_x &= \lim_{r\to\infty}\{8\pi\mu r^3(u\cos\theta-v\sin\theta)/(r^2+x^2)\}.\end{aligned}\right\} \tag{20}$$

Carrying out the indicated limits results in two separate expressions for F_x:

$$F_x = \sqrt{2}\pi\mu Uc\sum_{n=0}^{\infty}(4n^2-1)B_n = 4\sqrt{2}\pi\mu Uc\sum_{n=0}^{\infty}C_n \tag{21}$$

which are consistent with (19). Having alternate expressions for the force to some extent permits a check on the accuracy of our numerical calculations.

A dimensionless force coefficient f_x can be defined as the ratio of F_x to the force acting on a sphere of radius $(b+2a)$ translating with velocity U; the length $(b+2a)$ is chosen because it results in values of f_x which do not vary greatly even for large changes in b/a and therefore are suitable for interpolation.

$$f_x = \frac{F_x}{6\pi\mu U(b+2a)} = \frac{\sqrt{2}\sum_{n=0}^{\infty}(4n^2-1)B_n}{6(\coth\eta_0+\operatorname{cosech}\eta_0)} = \frac{2\sqrt{2}\sum_{n=0}^{\infty}C_n}{3(\coth\eta_0+\operatorname{cosech}\eta_0)}. \tag{22}$$

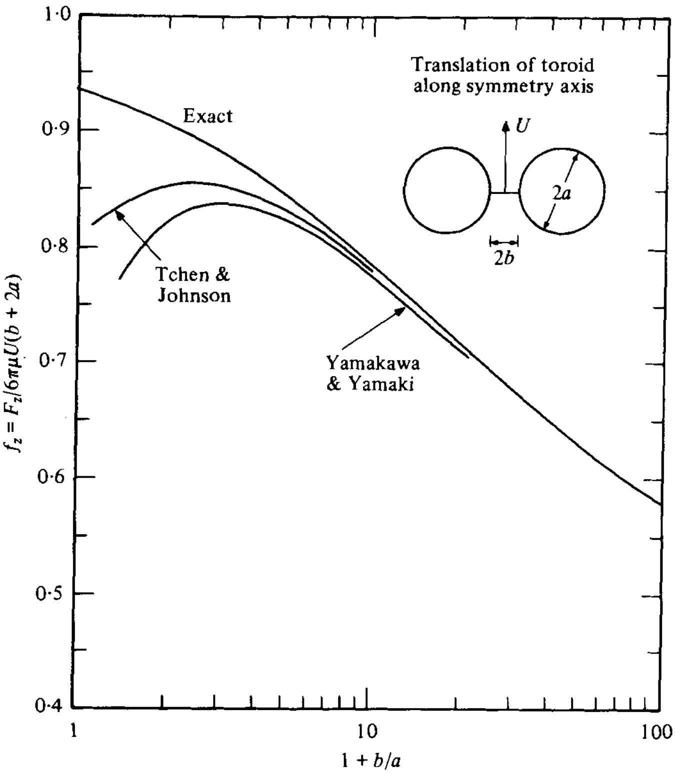

FIGURE 2. Comparison of exact and approximate dimensionless force coefficients for translation of a torus along its symmetry axis.

Calculated values of f_x are listed in table 1 for a range of values of $1+b/a = \cosh\eta_0$ from unity to 100. These are also shown in figure 1 where they are compared with the approximate calculation valid for large b/a discussed in the introduction.

Calculated values of f_z for translation along the axis of symmetry are included in table 1 and are shown in figure 2.

3. Rotation of a torus about a transverse axis

The calculation of the flow field for rotation resembles that given above for translation. For the case of rotation with angular velocity Ω about the y axis, the no-slip boundary condition requires that on the particle's surface $\mathbf{v} = \Omega z\mathbf{i}_x - \Omega x\mathbf{i}_z$ or equivalently in cylindrical co-ordinates

$$u = \Omega z\cos\theta; \quad v = -\Omega z\sin\theta; \quad w = -\Omega\tilde{\omega}\cos\theta. \tag{23}$$

We assume that the Reynolds number $\Omega a^2\rho/\mu$ is sufficiently small to allow us to neglect the nonlinear inertia terms in the Navier–Stokes equations. Accordingly, the equations governing the flow are equations (1) and (2). As with the case for translation, guided by the dependence of the velocity components on θ required by the boundary conditions, we again seek solutions of the form given by (4); U should be replaced by Ωc and the four new functions $\tilde{Q}$, $\tilde{U}$, $\tilde{V}$ and $\tilde{W}$ while still satisfying (5), (6) and (7) differ from the corresponding functions for translation. In fact, the symmetry of the rotational problem requires that u, and v be odd functions of z while w is an even function of z. In toroidal co-ordinates, $\tilde{Q}$, $\tilde{U}$ and $\tilde{V}$ are odd functions of ξ while $\tilde{W}$ is an even function of ξ. The solutions to (5) with the appropriate symmetry are

$$\left.\begin{aligned}
\tilde{W} &= (\cosh\eta-\cos\xi)^{\frac{1}{2}} \sum_{n=0}^{\infty} A_n P^1_{n-\frac{1}{2}}(\cosh\eta)\cos n\xi,\\
\tilde{Q} &= (\cosh\eta-\cos\xi)^{\frac{1}{2}} \sum_{n=1}^{\infty} B_n P^1_{n-\frac{1}{2}}(\cosh\eta)\sin n\xi,\\
\tilde{U} &= (\cosh\eta-\cos\xi)^{\frac{1}{2}} \sum_{n=1}^{\infty} C_n P_{n-\frac{1}{2}}(\cosh\eta)\sin n\xi,\\
\tilde{V} &= (\cosh\eta-\cos\xi)^{\frac{1}{2}} \sum_{n=1}^{\infty} D_n P^2_{n-\frac{1}{2}}(\cosh\eta)\sin n\xi,
\end{aligned}\right\} \tag{24}$$

where A_n, B_n, C_n and D_n are new coefficients different from those calculated for translation. It will be noted that these forms for $\tilde{Q}$, $\tilde{U}$, $\tilde{V}$ and $\tilde{W}$ automatically satisfy the boundary condition of vanishing velocity at infinity.

Substituting equations (4) with U replaced by Ωc into equations (24) gives, after a little rearrangement, the following conditions to be satisfied on the surface of the particle:

$$\tilde{U}-\tilde{V} = -2z/c, \quad z\tilde{U}-\tilde{\omega}\tilde{W} = \tilde{\omega}^2/c, \quad 2\tilde{U}+\tilde{\omega}\tilde{Q}/c = 0. \tag{25}$$

Use of the expansion in equation (11) and its derivative with respect to ξ, namely

$$\sin\xi/(\cosh\eta-\cos\xi)^{\frac{1}{2}} = (4\sqrt{2}/\pi)\sum_{n=1}^{\infty} Q_{n-\frac{1}{2}}(\cosh\eta)\sin n\xi \tag{26}$$

converts these three boundary conditions into the following:

$$\begin{gathered}
C_n P_{n-\frac{1}{2}} = D_n P^2_{n-\frac{1}{2}} + (8\sqrt{2}/\pi)\, nQ_{n-\frac{1}{2}} \quad (n \geqslant 1),\\
A_n 2\sinh\eta_0 P^1_{n-\frac{1}{2}} = -D_{n-1}P^2_{n-\frac{3}{2}}\zeta_n + D_{n+1}P^2_{n+\frac{1}{2}} + (8\sqrt{2}/\pi)\sinh\eta Q^1_{n-1} \quad (n \geqslant 0),\\
B_n \sinh\eta_0 P^1_{n-\frac{1}{2}} = D_{n-1}P^2_{n-\frac{3}{2}}\zeta_n - D_n 2\cosh\eta_0 P^2_{n-\frac{1}{2}} + D_{n+1}P^2_{n+\frac{1}{2}} \quad (n \geqslant 1),
\end{gathered} \tag{27}$$

where ζ_n is defined by (15) and again the argument of all the Legendre functions is $\cosh\eta_0$. The fourth relationship between the coefficients is found by substituting the series expansions of (24) into the continuity equation as expressed by (7). The result after much tedious algebra is

$$\begin{aligned}
&-(n-\tfrac{3}{2})A_{n-1}\delta_n + 2nA_n - (n+\tfrac{3}{2})A_{n+1} - \tfrac{1}{2}(n-\tfrac{3}{2})B_{n-1}\zeta_n\\
&+\tfrac{5}{2}B_n + \tfrac{1}{2}(n+\tfrac{3}{2})B_{n+1} - (\tfrac{1}{2})C_{n-1}\zeta_n + C_n - (\tfrac{1}{2})C_{n+1}\\
&-\tfrac{1}{2}(n-\tfrac{5}{2})(n-\tfrac{3}{2})D_{n-1}\zeta_n + (n-\tfrac{3}{2})(n+\tfrac{3}{2})D_n\\
&-\tfrac{1}{2}(n+\tfrac{3}{2})(n+\tfrac{5}{2})D_{n+1} = 0 \quad (n \geqslant 1)
\end{aligned} \tag{28}$$

where δ_n is given by (13). On substituting (27) into (28) we obtain a second-order difference equation for the coefficients D_n, namely

$$\begin{aligned} &D_{n-1}\zeta_n\{(n-\tfrac{3}{2})(X_{n-1}+Y_{n-1}-\tfrac{3}{2})+(5-2n)X_n-(\tfrac{1}{2})Z_{n-1}-(\tfrac{1}{2})(n-\tfrac{5}{2})(n-\tfrac{3}{2})\} \\ &\quad +D_n\{-2(n-\tfrac{3}{2})Y_{n-1}-5(X_n+Y_n-\tfrac{3}{2})+2(n+\tfrac{3}{2})X_{n+1}+Z_n+(n-\tfrac{3}{2})(n+\tfrac{3}{2})\} \\ &\quad +D_{n+1}\{(2n+5)Y_n-(n+\tfrac{3}{2})(X_{n+1}+Y_{n+1}-\tfrac{3}{2})-(\tfrac{1}{2})Z_{n+1}-(\tfrac{1}{2})(n+\tfrac{3}{2})(n+\tfrac{5}{2})\} \\ &= (n-1)V_{n-1}-2nV_n+(n+1)V_{n+1}+(n-\tfrac{3}{2})U_{n-1}-2nU_n+(n+\tfrac{3}{2})U_{n+1}, \end{aligned} \tag{29}$$

for $n \geqslant 1$ where V_n, X_n, Y_n and Z_n are as defined in equations (17) and

$$U_n \equiv 4\sqrt{2}Q^1_{n-\frac{1}{2}}/\pi P^1_{n-\frac{1}{2}}, \tag{30}$$

with the argument of all Legendre functions again being $\cosh \eta_0$.

The coefficients D_n must converge to zero as n increases to infinity, which permits the required solution to be found by successive truncation to a finite set of linear equations. Once the values of D_n are known, A_n, B_n and C_n are readily computed from equations (27). In carrying out these numerical calculations sufficient terms have been retained for each value of b/a so that the calculated torques are correct to at least four significant figures.

The torque $\mathbf{G}$ acting on the torus is given by

$$\mathbf{G} = \int_s (\mathbf{r}\times\mathbf{R}_n).\,d\mathbf{S} \tag{31}$$

when the moments of the surface stress $\mathbf{R}_n$ are taken about the origin $\tilde{\omega} = z = 0$. Substituting the derived expressions for the pressure and velocity gradients evaluated at the surface of the torus and carrying out the indicated integrations shows that the resisting torque has Cartesian components $(0, -G_y, 0)$ with

$$G_y = \sqrt{2}\pi\mu\Omega c^3 \sum_{n=0}^{\infty}\{(4n^2-1)A_n - 4nC_n\}. \tag{32}$$

The details are not given here because it is possible to derive expressions for the torque much more simply from the far-field expressions for the fluid velocity. At a great distance from the torus the flow must approach that associated with a point couple G_y in the y direction applied at the origin. This flow field is given by Happel & Brenner (1965) in Cartesian co-ordinates as $(G_y/8\pi\mu r^3)(z, 0, -x)$. Thus G_y must also be given by the following limits as $r \to \infty$, i.e. as ξ and $\eta \to 0$.

$$\left.\begin{aligned} G_y &= \lim_{r\to\infty}\{8\pi\mu r^3(u\cos\theta - v\sin\theta)/z\}, \\ G_y &= \lim_{r\to\infty}\{-8\pi\mu r^3 w/x\}. \end{aligned}\right\} \tag{33}$$

Carrying out the indicated limits results in two expressions for the torque:

$$G_y = 2\sqrt{2}\pi\mu\Omega c^3 \sum_{n=0}^{\infty}(4n^2-1)A_n = -8\sqrt{2}\pi\mu\Omega c^3 \sum_{n=1}^{\infty} nC_n, \tag{34}$$

which are consistent with equation (32).

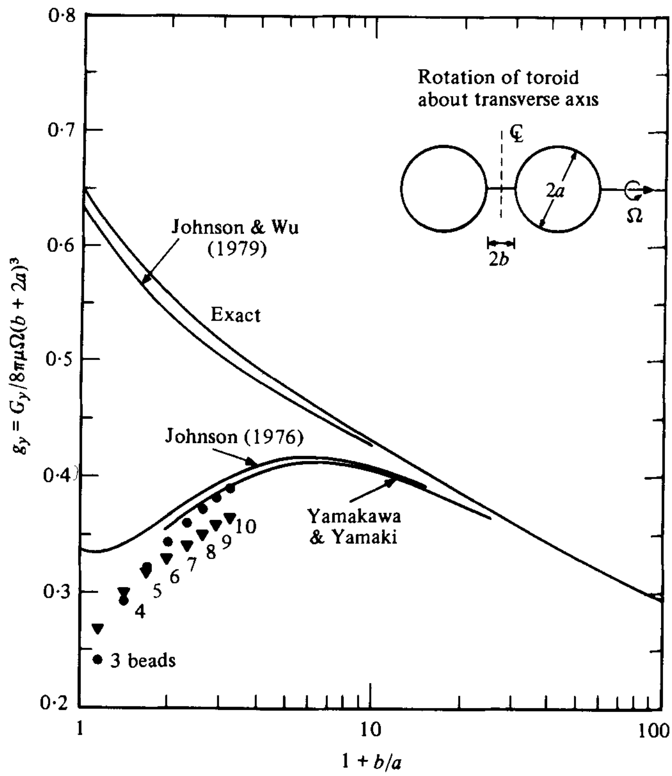

FIGURE 3. Comparison of exact and approximate dimensionless torque coefficients for rotation of a torus about a transverse axis. Bloomfield's results: ●, unmodified tensor; ▼, modified tensor.

A dimensionless torque coefficient g_y can be defined as the ratio of G_y to the torque acting on a sphere of radius $(b+2a)$ rotating with angular velocity Ω; again the length $(b+2a)$ is chosen because it results in values of g_y which do not vary greatly even for large changes in b/a and therefore are suitable for interpolation.

$$g_y = \frac{G_y}{8\pi\mu\Omega(b+2a)^3} = \frac{\sqrt{2}\sum_{n=0}^{\infty}(4n^2-1)A_n}{4(\coth\eta_0+\operatorname{cosech}\eta_0)^3} = -\frac{\sqrt{2}\sum_{n=1}^{\infty}nC_n}{(\coth\eta_0+\operatorname{cosech}\eta_0)^3}. \tag{35}$$

Calculated values of g_y are listed in table 1 for a range of values of $1+b/a = \cosh\eta_0$ from unity to 100. These are also shown in figure 3.

Calculated values of g_z for rotation about the axis of symmetry are included in table 1 and are shown in figure 4.

4. Discussion

Numerical values of the dimensionless force coefficient $f_x = F_x/6\pi\mu U(b+2a)$ for translation along a transverse axis and the dimensionless torque coefficient

$$g_y = G_y/8\pi\mu\Omega(b+2a)^3$$

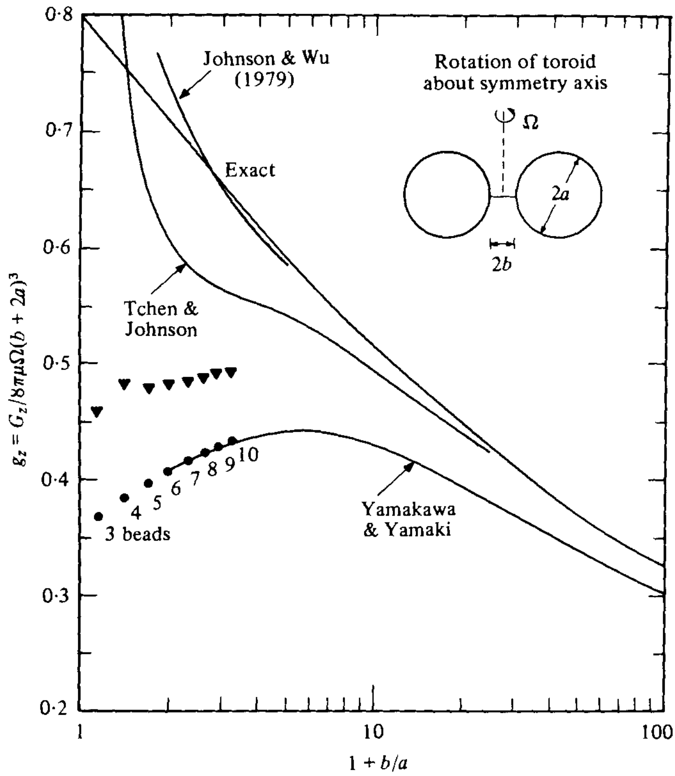

FIGURE 4. Comparison of exact and approximate dimensionless torque coefficients for rotation of a torus about its symmetry axis. Bloomfield's results: ●, unmodified; ▼, modified tensor.

for rotation about a transverse axis were computed from equations (21) and (35) respectively; these are listed in table 1. For convenience we have also listed in table 1 values of the force coefficient $f_z = F_z/6\pi\mu U(b+2a)$ for translation along the symmetry axis and values of the torque coefficient $g_z = G_z/8\pi\mu\Omega(b+2a)^3$ for rotation about the symmetry axis.

In figures 1–4 we compare these exact values with the approximations discussed in the introduction. It is seen that the exact and approximate values become asymptotically equal as b/a becomes very large but departures from the exact calculations are apparent for b/a less than about 10–100, depending on the mode of motion and the method of approximation. Poorest agreement is with the bead calculations of Yamakawa & Yamaki (1973) for rotation about the symmetry axis. This may be explained because for rotation about the symmetry axis, each bead moves in the wake of an adjacent bead. We know from the exact solution to the equations of creeping motion for equal touching spheres that the correction to Stokes law is greater for two spheres moving along their line of centres than it is for motion normal to the line of centres. Moreover, the first term correction to Stokes law for touching spheres moving along their line of centres as given by the method of reflexions (which is equivalent to use of the unmodified Oseen tensor in the bead calculations) is 0·571 compared to the exact value of 0·645, or an 11 % difference; the error will be compounded in a chain because each sphere has two immediate neighbours as well as more remote neighbours. The first term correction by the method of reflexions to Stokes law for touching spheres moving normal to the line of centres is 0·727 compared to the exact value of 0·716, or only a 1·5 % difference. Thus it is not surprising that the bead calculation is poorest for rotation about the symmetry axis. Use of the modified Oseen tensor shifts the calculations in the correct direction; however the shift is not adequate to achieve correspondence with the exact values, at least for small numbers of beads.

Best agreement between the bead calculations and the exact values occurs for the translation along the symmetry axis. This is consistent with the above discussion because for this motion, the beads do not move in wakes behind other beads. Intermediate cases are rotation about a transverse axis and translation along a transverse axis. In the former case, for which the calculations of Bloomfield (1976, private correspondence to Professor C. T. O'Konski) are available, use of the modified Oseen tensor results in poorer agreement with the exact values for five or more beads.

The slender body theory of Johnson (1976) and Tchen (1954) is somewhat more accurate than the bead calculations except in two cases where the formulae diverge for b/a on the order of unity. When the next level of slender-body calculation, as reported by Johnson & Wu (1979), in which terms of $O(a/b)^2$ are retained, is used, figures 3 and 4 show excellent agreement with the exact calculation for b/a as small as 2·0.

We note in passing that $g_y/g_z \to 1$ as $b/a \to \infty$ so that a long slender ring has the same resistance to rotation about its symmetry axis as it has to rotation about a transverse axis. The ratio f_z/f_x approaches $\frac{4}{3}$ as $b/a \to \infty$. These limiting ratios were established by using asymptotic expansions of the Legendre functions for large η (i.e. large b/a) and then retaining only the leading terms in the infinite sums for f_x, f_z, g_y and g_z. These limits can also be verified using the asymptotic formulae of Johnson & Wu (1979).

S. L. Goren and C. T. O'Konski are currently exploring the application of toroidal models to acetylcholine receptor, low molecular weight circular DNA, and other macromolecules.

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